

Type Classes for Efficient Exact Real Arithmetic in Coq

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 - ▶ Numerical algorithms in research papers.
 - ▶ Actual implementations (MATHEMATICA, MATLAB, ...).

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- ▶ **Makes it difficult to trust the code of these implementations!**
- ▶ Undesirable in proofs that rely on the execution of this code.
 - ▶ Kepler conjecture.
 - ▶ Existence of the Lorentz attractor.
- ▶ Undesirable in safety critical applications.

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Improve performance of real number computation in Coq.

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- ▶ Approximation by rational numbers.
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Coq:

- ▶ Well suited because it is both a dependently typed functional programming language, and,
- ▶ a proof assistant for constructive mathematics.

Starting point: O'Connor's implementation in Coq

- ▶ Based on *metric spaces* and the *completion monad*.

$$\mathbb{R} := \mathfrak{C}\mathbb{Q} := \{f : \mathbb{Q}_+ \rightarrow \mathbb{Q} \mid f \text{ is regular}\}$$

- ▶ To define a function $\mathbb{R} \rightarrow \mathbb{R}$: define a *uniformly continuous function* $f : \mathbb{Q} \rightarrow \mathbb{R}$, and obtain $\check{f} : \mathbb{R} \rightarrow \mathbb{R}$.
- ▶ Efficient combination of proving and programming.

O'Connor's implementation in Coq

Problem:

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Solution:

Build theory and programs on top of **abstract interfaces** instead of concrete implementations.

- ▶ Cleaner.
- ▶ Mathematically sound.
- ▶ Can swap implementations.

Our contribution

An abstract specification of the dense set.

- ▶ For which we provide an implementation using the dyadics:

$$n * 2^e \quad \text{for} \quad n, e \in \mathbb{Z}$$

- ▶ Using CoQ's machine integers.
- ▶ Extend the algebraic hierarchy based on type classes by Spitters and van der Weegen to achieve this.

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- ▶ Extend the algebraic hierarchy based on type classes by Spitters and van der Weegen to achieve this.

Some other performance improvements.

- ▶ Implement range reductions.
- ▶ Improve computation of power series:
 - ▶ Keep auxiliary results small.
 - ▶ Avoid evaluation of termination proofs.

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Type class based interfaces for:

- ▶ A standard algebraic hierarchy.
- ▶ Some category theory.
- ▶ Some universal algebra.
- ▶ Interfaces for number structures.
 - ▶ Naturals: initial semiring.
 - ▶ Integers: initial ring.
 - ▶ Rationals: field of fractions of \mathbb{Z} .

Our extensions of Spitters and van der Weegen

- ▶ Interfaces and theory for operations (`nat_pow`, `shifl`, ...).
- ▶ Support for undecidable structures.
- ▶ Library on constructive order theory (ordered rings, etc...)
- ▶ Explicit casts.

Support for undecidable structures

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Support for undecidable structures

- ▶ To compute $\frac{1}{x}$ for $x \in \mathbb{R}$, one needs a witness $\varepsilon \in \mathbb{Q}_+$ such that $|x| \geq \varepsilon$.
- ▶ Cannot be extracted from a proof of $x \neq 0$ because a negation lacks computational content.
- ▶ Need **apartness** \lesssim instead of inequality.
 1. $\neg x \lesssim x$ (irreflexive)
 2. $x \lesssim y \rightarrow y \lesssim x$ (symmetric)
 3. $x \lesssim y \rightarrow (x \lesssim z \vee y \lesssim z)$ (co-transitive)
 4. $\neg x \lesssim y \leftrightarrow x = y$ (tight)

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- ▶ **Very heavy in practice.**

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- ▶ Requires additional work to extract witnesses.
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- ▶ Use type classes to reduce bookkeeping.
- ▶ Easier in practice.

Extracting witnesses

Use constructive indefinite description

Lemma `constructive_indefinite_description_nat` ($P : \text{nat} \rightarrow \text{Prop}$) :
 $(\forall x : \text{nat}, \{P x\} + \{\neg P x\}) \rightarrow (\exists n : \text{nat}, P n) \rightarrow \{n : \text{nat} \mid P n\}$

to extract a witness from a `Prop`-based apartness.

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- ▶ Performs linear bounded search.
Slow!
- ▶ We specify explicit witnesses for computation.
Faster to obtain, better quality.

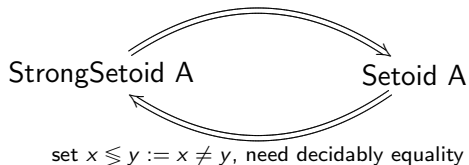
Cyclic instances

- ▶ We have to look out for cyclic instances, for example



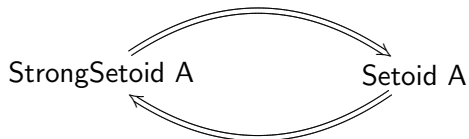
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set $x \leq y := x \neq y$, need decidable equality

makes instance search loop.

- ▶ Create `StrongSetoid A` from `Setoid A` instances by hand.

Approximate rationals

Class AppDiv AQ := app_div : AQ → AQ → Z → AQ.

Class AppApprox AQ := app_approx : AQ → Z → AQ.

Class AppRationals AQ {e plus mult zero one inv} '{!Order AQ}
{AQtoQ : Coerce AQ Q_as_MetricSpace} '{!AppInverse AQtoQ}
{ZtoAQ : Coerce Z AQ} '{!AppDiv AQ} '{!AppApprox AQ}
'{!Abs AQ} '{!Pow AQ N} '{!ShiftL AQ Z}
'{∀ x y : AQ, Decision (x = y)} '{∀ x y : AQ, Decision (x ≤ y)} : Prop := {
aq_ring :> @Ring AQ e plus mult zero one inv ;
aq_order_embed :> OrderEmbedding AQtoQ ;
aq_ring_morphism :> SemiRing_Morphism AQtoQ ;
aq_dense_embedding :> DenseEmbedding AQtoQ ;
aq_div : ∀ x y k, \mathbf{B}_{2^k} (app_div x y k) ('x / 'y) ;
aq_approx : ∀ x k, \mathbf{B}_{2^k} (app_approx x k) ('x) ;
aq_shift :> ShiftLSpec AQ Z (<<);
aq_nat_pow :> NatPowSpec AQ N (^) ;
aq_ints_mor :> SemiRing_Morphism ZtoAQ }.

Creating the real numbers

- ▶ Show that the approximate rationals form a metric space.
- ▶ Complete it to obtain the real numbers.
- ▶ Lift the ring operations to the real numbers.
- ▶ Prove correspondence with O'Connor's implementation.

Power series

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 - ▶ its coefficients are alternating,
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 - ▶ its coefficients are alternating,
 - ▶ decreasing,
 - ▶ and have limit 0.
- ▶ For example, for $-1 \leq x \leq 1$:

$$\sin x = \sum_{i=0}^{\infty} (-1)^i * \frac{x^{2i+1}}{2i+1}$$

- ▶ To approximate $\sin x$ with error ε we find a k such that:

$$\left| (-1)^i * \frac{x^{2i+1}}{2i+1} \right| \leq \varepsilon$$

Power series

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- ▶ This can be optimized using shifts.

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- ▶ So, we cannot compute the coefficients $\frac{x^{2i+1}}{2i+1}$ exactly.
- ▶ Use 2 streams: numerators and denominators.
- ▶ Need to compute both the length and precision of division.
- ▶ This can be optimized using shifts.
- ▶ Our approach only requires to compute few extra terms.
- ▶ Approximate division keeps the auxiliary numbers “small” .

Power series

Problem 2: convince `Coq` that it terminates.

- ▶ Use an inductive proposition to describe limits.

Inductive `Exists A (P : Stream A → Prop) (x : Stream) : Prop :=`
| `Here : P x → Exists P x`
| `Further : Exists P (tl x) → Exists P x.`

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| `Here : P x → Exists P x`
| `Further : Exists P (tl x) → Exists P x.`

- ▶ But, need to make it lazy, otherwise `vm_compute` will evaluate a proposition [O'Connor].

Inductive `LazyExists A (P : Stream A → Prop) (x : Stream A) : Prop :=`
| `LazyHere : P x → LazyExists P x`
| `LazyFurther : (unit → LazyExists P (tl x)) → LazyExists P x.`

Power series

Unfortunately, still too much overhead.

- ▶ Perform 50.000 steps before looking at the proof.

```
Fixpoint LazyExists_inc '{P : Stream A → Prop}
  (n : nat) s : LazyExists P (Str_nth_tl n s) → LazyExists P s :=
  match n return LazyExists P (Str_nth_tl n s) → LazyExists P s with
  | 0 ⇒ λ x, x
  | S n ⇒ λ ex, LazyFurther (λ _, LazyExists_inc n (tl s) ex)
  end.
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- ▶ Major (≥ 10 times) performance improvement!

Extending the sine to its complete domain

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$$\sin x = 3 * \sin \frac{x}{3} - 4 * \left(\sin \frac{x}{3} \right)^3$$

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- ▶ Efficient because we postpone divisions.
- ▶ Performance improves significantly by reducing the input to a value between $-2^k \leq x \leq 0$ for $50 \leq k$.
- ▶ Faster than subtracting multiples of 2π because our implementation of π is too slow.

What have we implemented so far?

Verified versions of:

- ▶ Basic field operations (+, *, -, /)
- ▶ Exponentiation by a natural.
- ▶ Computation of power series.
- ▶ exp, arctan, sin and cos.
- ▶ $\pi := 176 * \arctan \frac{1}{57} + 28 * \arctan \frac{1}{239} - 48 * \arctan \frac{1}{682} + 96 * \arctan \frac{1}{12943}$.
- ▶ Square root using Wolfram iteration.

Benchmarks

- ▶ Our HASKELL prototype is ~ 15 times faster.
- ▶ Our COQ implementation is ~ 100 times faster.
- ▶ For example:
 - ▶ 500 decimals of $\exp(\pi * \sqrt{163})$ and $\sin(\exp 1)$,
 - ▶ 2000 decimals of $\exp 1000$,within 10 seconds in COQ!
- ▶ (Previously about 10 decimals)

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 - ▶ 2000 decimals of $\exp 1000$,within 10 seconds in COQ!
- ▶ (Previously about 10 decimals)
- ▶ Type classes only yield a 3% performance loss.
- ▶ COQ is still too slow compared to unoptimized HASKELL (factor 30 for Wolfram iteration).

Further work

- ▶ Newton iteration to compute the square root.
- ▶ Geometric series (e.g. to compute \ln).
- ▶ `native_compute`: evaluation by compilation to OCAML.
- ▶ FLOCQ: more fine grained floating point algorithms.
- ▶ Type classified theory on metric spaces.

Conclusions

- ▶ Greatly improved the performance of the reals.
- ▶ Abstract interfaces allow to swap implementations and share theory and proofs.
- ▶ Type classes yield no apparent performance penalty.
- ▶ Nice names and notations with type classes and unicode symbols.

Issues

- ▶ Type classes are quite fragile.
- ▶ Instance resolution is too slow.
- ▶ Instance resolution cannot handle cyclic instances.
- ▶ No setoid rewriting in for relations in `Type`.
- ▶ Need to adapt definitions to avoid evaluation in `Prop`.

Sources

<http://robbertkrebbers.nl/research/realis/>