

# A call-by-value $\lambda$ -calculus with lists and control

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Warwick, United Kingdom

# Problem

- ▶ A lot of  $\lambda$ -calculi with control have been studied:  
 $\lambda_C$ ,  $\lambda_\mu$ ,  $\bar{\lambda}_\mu\tilde{\mu}$ ,  $\lambda_{C^-}$ ,  $\lambda_\Delta$ , and many more...

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- ▶ Actual programming languages do support these

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  - ▶ call-by-name reduction with call-by-value data types
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  - ▶ hard to extend

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- ▶ However:
  - ▶ No types like natural numbers, lists, ...
  - ▶ No (direct) proofs of confluence and strong normalization

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- ▶ These properties are relatively easy to prove

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- ▶ Another way to think of it:  $t$  is a proof of either
  - ▶  $\rho$ , or,
  - ▶  $\alpha : \psi \in \Delta$

## The typing rules of $\lambda::\text{catch}$

The constructs of simple type theory:

$$\frac{x : \rho \in \Gamma}{\Gamma; \Delta \vdash x : \rho} \quad \frac{\Gamma, x : \sigma; \Delta \vdash t : \tau}{\Gamma; \Delta \vdash \lambda x. t : \sigma \rightarrow \tau} \quad \frac{\Gamma; \Delta \vdash t : \sigma \rightarrow \tau \quad \Gamma; \Delta \vdash s : \sigma}{\Gamma; \Delta \vdash ts : \tau}$$

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Constructors of the unit and list data type:

$$\overline{\Gamma; \Delta \vdash () : \top} \quad \overline{\Gamma; \Delta \vdash \text{nil} : [\sigma]} \quad \overline{\Gamma; \Delta \vdash (::) : \sigma \rightarrow [\sigma] \rightarrow [\sigma]}$$

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Primitive recursion over lists:

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**Important:**  $\psi$  ranges over  $\rightarrow$ -free types [Herbelin, 2010]

## Example: typing

---

`; ⊢ catch α . (throw α nil) :: nil : [T]`

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How to think of this derivation:

1. Our goal is  $[\top]$



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4. But instead we jump back to  $\alpha$  with  $\text{nil}$

# Reduction

## Values:

$$v, w ::= x \mid () \mid \text{nil} \mid (::) \mid (::) v \mid (::) v w \\ \mid \text{lrec} \mid \text{lrec } v_r \mid \text{lrec } v_r v_s \mid \lambda x. r$$

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$$E ::= \square t \mid v \square \mid \text{throw } \beta \square$$

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**Progress:**  $;\vdash t : \tau \implies t$  is a value or  $\exists t'. t \rightarrow t'$

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- ▶ Hence progress would fail
- ▶ Note: an analogue term in the  $\lambda\mu$ -calculus

$$\mu \alpha . [\alpha] \lambda x . \mu \_ . [\alpha] \lambda y . y$$

does not reduce either

## Why restrict to $\rightarrow$ -free types? (2)

### Consequences of progress

- ▶ In Herbelin's  $\text{IQC}_{\text{MP}}$ :
  - ▶ If  $; \vdash t : \rho \vee \sigma$ , then  $\exists t' . ; \vdash t' : \rho$  or  $; \vdash t' : \sigma$
  - ▶ If  $; \vdash t : \exists x.P(x)$ , then  $\exists t' . ; \vdash t' : P(t')$

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  - ▶ If  $; \vdash t : \exists x. P(x)$ , then  $\exists t' . ; \vdash t' : P(t')$
- ▶ In  $\lambda::\text{catch}$ :
  - ▶ Unique representation of data
  - ▶ One-to-one correspondence between closed terms of  $\mathbb{N}$  and  $\mathbb{N}$



# Natural numbers

We define a type  $\mathbb{N} := [\mathbb{T}]$ , with:

$$0 := \text{nil}$$
$$S := (::) ()$$
$$\text{nrec} := \lambda x_r y_s . \text{lrec } x_r (\lambda \_ . x_s)$$

Notation:  $\underline{n} := S^n 0$

## Inefficient predecessor

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$$\text{pred } \underline{n} \rightarrow (\lambda x h . x) \underline{n - 1} (\text{pred } \underline{n - 1})$$

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$$\begin{aligned} \text{pred } \underline{n} &\rightarrow (\lambda x h . x) \underline{n-1} (\text{pred } \underline{n-1}) \\ &\rightarrow (\lambda h . \underline{n-1}) \underbrace{(\text{pred } \underline{n-2})}_{\text{not a value}} \end{aligned}$$

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## Example: list multiplication

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- ▶ We use control to jump out when we encounter a zero

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## Properties of $\lambda::\text{catch}$

- ▶ **Subject reduction.**

$\Gamma; \Delta \vdash t : \rho$  and  $t \rightarrow t'$ , then  $\Gamma; \Delta \vdash t' : \rho$

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- ▶ **Strong Normalization.**

$\Gamma; \Delta \vdash t : \rho$ , then no infinite  $t \rightarrow t_1 \dots$

## Parallel reduction

Usual approach [Tait/Martin-Löf]

1. Define a parallel reduction  $\Rightarrow$
2. Prove that  $\Rightarrow$  is confluent
3. Prove that  $t_1 \rightarrow t_2$  implies  $t_1 \Rightarrow t_2$
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For the ordinary  $\lambda$ -calculus

$$\frac{}{x \Rightarrow x} \quad \frac{t \Rightarrow t'}{\lambda x. t \Rightarrow \lambda x. t'} \quad \frac{t \Rightarrow t' \quad r \Rightarrow r'}{tr \Rightarrow t'r'}$$
$$\frac{t \Rightarrow t' \quad r \Rightarrow r'}{(\lambda x. t)r \Rightarrow t'[x := r']}$$

## Parallel reduction for $\lambda::\text{catch}$

- ▶ Consider the naive rule for `throw`

$$\frac{t \Rightarrow t'}{E[\text{throw } \alpha t] \Rightarrow \text{throw } \alpha t'}$$

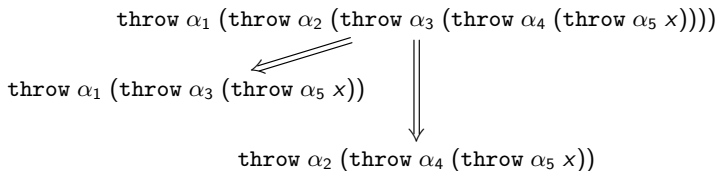


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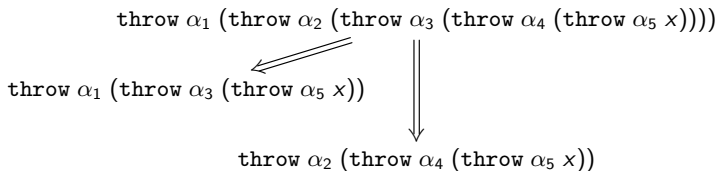


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- ▶ **Solution:** jump over a compound context

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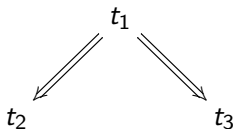
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Define  $t^\diamond$  such that if  $t_1 \Rightarrow t_2$ , then  $t_2 \Rightarrow t_1^\diamond$  [Takahashi, 1995]

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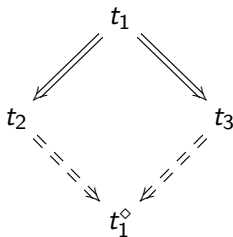
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## Complete development for $\lambda::\text{catch}$

$$((\lambda x.t) v)^\diamond := t^\diamond[x := v^\diamond]$$

$$(\vec{E}[\text{throw } \alpha t])^\diamond := \text{throw } \alpha t^\diamond \text{ if } t \not\equiv \text{throw } \gamma s$$

$$(\text{catch } \alpha . \text{throw } \alpha t)^\diamond := \text{catch } \alpha . t^\diamond$$

$$(\text{catch } \alpha . \text{throw } \beta v)^\diamond := \text{throw } \beta v^\diamond \text{ if } \alpha \notin \{\beta\} \cup \text{FCV}(v)$$

$$(\text{catch } \alpha . v)^\diamond := v^\diamond \quad \text{if } \alpha \notin \text{FCV}(v)$$

$$(\text{lrec } v_r v_s \text{ nil})^\diamond := v_r^\diamond$$

$$(\text{lrec } v_r v_s (v_h :: v_t))^\diamond := v_s^\diamond v_h^\diamond v_t^\diamond (\text{lrec } v_r^\diamond v_s^\diamond v_t^\diamond)$$

...

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- ▶ If  $x_1 : \rho_1, \dots, x_n : \rho_n; \Delta \vdash t : \tau$  and  $r_i \in \llbracket \rho_i \rrbracket$  for all  $1 \leq i \leq n$ , then  $t[x_1 := r_1, \dots, x_n := r_n] \in \llbracket \tau \rrbracket$

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- ▶ Program extraction à la Paulin/Letouzey